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LETTER TO THE EDITOR

Generalization of Weierstrassian elliptic functions to $\mathbf{R}^n$

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Abstract. The Weierstrassian $\wp$, ζ and σ functions are generalized to $\mathbf{R}^n$. The $n = 3$ and $n = 4$ cases have already been used in instanton solutions which may be interpreted as explicit realizations of spacetime foam and the monopole condensate, respectively. The new functions satisfy higher dimensional versions of the periodicity properties and Legendre's relations obeyed by their familiar complex counterparts. For $n = 4$, the construction reproduces functions found earlier by Fueter using quaternionic methods. Integrating over lattice points along all directions but two, one recovers the original Weierstrassian elliptic functions.

Kaluza–Klein, supergravity, string and superstring theories and super p-branes all involve dimensions beyond four, which then have to be compactified. The compactification results in lattices in the higher dimensions [1]. It might therefore be of interest to consider generalizations of doubly periodic functions to arbitrary $\mathbf{R}^n$. The $\mathbf{R}^4$ case has already been treated by Fueter [2], who succeeded in obtaining quaternionic analogues of Weierstrassian elliptic functions. In this letter, we will present a unique and straightforward extension of Fueter's results to $\mathbf{R}^n$. This general construction yields Fueter's functions in $n = 4$ without requiring the use of quaternions. Furthermore, when one takes infinitesimal lattice spacings in all but two lattice directions and integrates over these points one recovers the usual Weierstrassian functions.

Apart from possible future physics applications of n-tuply periodic functions, it is worth noting that (quasi)periodic classical solutions of gravity and Yang–Mills theory have already been considered. Thus, inspired by Rossi's observation [3] that a singly periodic instanton configuration represents a BPS monopole [4], Gursey and Tze [5] used Fueter's hyperelliptic functions to construct a solution with one Yang–Mills instanton per spacetime cell. This solution may be related to the monopole condensate [6] or to the Copenhagen vacuum [7], which is also based on a periodic arrangement of the theory's solitons. Hawking, on the other hand, argued [8] that metrics with a single gravitational instanton per unit cell make the dominant contribution to the path integral of Einstein's gravity. An adaptation of Weierstrassian σ and ζ functions to three dimensions [9] along the lines presented here leads to just such a self-dual solution, providing an explicit example of a 'spacetime foam' based on Gibbons–Hawking multicentre metrics [10].

We start by examining how the infinite sums defining the Weierstrassian functions are made to converge. These are

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\neq 0}\left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\} \tag{1}$$

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$$\zeta(z) = -\int \wp(z)\,\mathrm{d}z = \frac{1}{z} + \sum_{\omega\neq 0}\left\{\frac{1}{(z-\omega)} + \frac{1}{\omega} + \frac{z}{\omega^2}\right\} \tag{2}$$

$$\rho(z) \equiv \ln\sigma(z) = \int \,\mathrm{d}z\,\zeta(z) = \ln(z) + \sum_{\omega\neq 0}\left\{\ln(z-\omega) - \ln(-\omega) + \frac{z}{\omega} + \frac{z^2}{2\omega^2}\right\} \tag{3}$$

or,

$$\rho(z) = \ln(z) + \sum_{\omega\neq 0}\left\{\ln\left(1-\frac{z}{\omega}\right) + \frac{z}{\omega} + \frac{z^2}{2\omega^2}\right\}. \tag{4}$$

In the above $\omega = n_1\omega_1 + n_2\omega_2$, with (ω_1, ω_2) defining the basic lattice cell $\mathcal{C}$. The parentheses $\{\}$ ensure that each term in the sum is absolutely convergent; the series become meaningless if the parentheses are broken up and the terms they contain are separately summed. An efficient method for studying the convergence properties of (1)–(3) is provided by an integral test in which the double sum is replaced by $\int_{|\omega|_{\min}}^{\infty} \mathrm{d}|\omega|\,|\omega|$. This reveals why one, two and three subtraction terms are needed in (1), (2) and (3), respectively. Employing a simple dimensional argument, one can proceed further and determine the precise forms of the subtraction terms, given only the first term in the sum (3), for example. To do this, we note (i) the ρ is dimensionless (assigning the dimension of length to the coordinate z, say); (ii) in the power series expansion of (4), the highest power of ω permitted by convergence requirements is ω^{-3}. One then realizes that the subtraction terms following $\ln(1 - z/\omega)$ must be chosen so that $\rho(z)$ has the expansion

$$\rho(z) = \ln z + \sum_{\omega\neq 0} \mathrm{O}\left(\frac{z^3}{\omega^3}\right) + \cdots . \tag{5}$$

Hence using the power series

$$\ln(1 - z/\omega) = -z/\omega - z^2/2\omega^2 - z^3/3\omega^3 - \cdots \tag{6}$$

one chooses the three subtraction terms

$$-\ln(-\omega) + z/\omega + z^2/2\omega^2 \tag{7}$$

to arrive at the form (5). This will be the key in constructing $\mathbf{R}^n$ analogues of $\rho(z)$.

We will later need the well known facts that $\wp(z)$ is doubly periodic while $\zeta(z)$ and $\sigma(z)$ are quasiperiodic with the transformation properties

$$\zeta(z + \omega_{1,2}) = \zeta(z) + \eta_{1,2} \tag{8}$$

and

$$\sigma(z + \omega_{1,2}) = -\sigma(z)\exp\eta_{1,2}(z + \omega_{1,2}/2) \tag{9}$$

where $\eta_{1,2} = 2\zeta(\omega_{1,2}/2)$.

We also record the so-called Legendre's relations

$$\oint_{\partial\mathcal{C}} \zeta(z)\,\mathrm{d}z = 2\pi\mathrm{i} \tag{10}$$

and

$$\oint_{\partial\mathcal{C}} \zeta(z)\,\mathrm{d}z = 2\pi\mathrm{i} = \eta_1\omega_2 - \eta_2\omega_1 . \tag{11}$$

We first rewrite (10) in the rather unconventional form reminiscent of Gauss's theorem

$$\int\!\!\int_{\mathcal{C}} \mathrm{d}V\,\nabla^2\rho(z) = \oint_{\partial\mathcal{C}} \mathrm{d}\vec{\sigma}\cdot\vec{\nabla}\rho(z) = 2\pi \tag{12}$$

where $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y$ and the 'surface element' $\mathrm{d}\vec{\sigma} = \hat{n}\,\mathrm{d}l$. Here $\mathrm{d}l$ is the arclength and $\hat{n}$ a unit normal vector pointing outwards on $\partial\mathcal{C}$.

Let us next introduce the lattice basis vectors $q_\mu^{(a)}$, where $a, \mu = 1, \ldots, n$. The volume of the unit cell $\mathcal{C}$ is given by

$$V_n = \frac{1}{n!}\epsilon_{a_1\ldots a_n}\epsilon^{\mu_1\ldots\mu_n} q_{\mu_1}^{(a_1)} \ldots q_{\mu_n}^{(a_n)} . \tag{13}$$

The basis vectors of the reciprocal lattice obey

$$r_\mu^{(a)} q_\mu^{(b)} \equiv r^{(a)} \cdot q^{(b)} = \delta^{ab} . \tag{14}$$

They are obtained from the $q_\mu^{(a)}$ via

$$r_{\mu_1}^{(a_1)} = \frac{1}{n!V_n}\epsilon_{a_1\ldots a_n}\epsilon^{\mu_1\ldots\mu_n} q_{\mu_2}^{(a_2)} \ldots q_{\mu_n}^{(a_n)}. \tag{15}$$

We now seek higher dimensional versions of (12) in the form

$$\int_{\mathcal{C}} \mathrm{d}V_n \partial_\mu \partial_\mu \rho_n(x) = \oint_{\partial\mathcal{C}} \mathrm{d}\sigma_\mu \partial_\mu \rho_n(x) = -\int \mathrm{d}\omega_n \tag{16}$$

where, of course,

$$\int \mathrm{d}\omega_n \equiv \Omega_n = 2\pi^{n/2}/\,\Gamma(n/2) . \tag{17}$$

In order to obtain the result (16), the function $\rho_n(x)$, which is to serve as the analogue of $\rho(z)$, should have the form

$$\rho_n(x) \propto G_n(x) + \sum \cdots \sum_{q\neq 0} G_n(x-q) + (\mathbf{R}^n \text{ harmonics}) . \tag{18}$$

In equation (18), $G_n(x)$ is the Green's function for $\mathbf{R}^n$ obeying

$$\partial_\mu \partial_\mu\, G_n(x) = -\Omega_n \delta(x_1) \ldots \delta(x_n) \equiv -\Omega_n \delta(x) \tag{19}$$

and

$$q = n_1 q^{(1)} + \cdots + n_n q^{(n)}. \tag{20}$$

We have, of course,

$$G_n(x-q) = \frac{1}{(x^2 - 2xq + q^2)^{\frac{n-2}{2}}} = \frac{1}{|x-q|^{n-2}} \qquad (n > 2) \tag{21}$$

where $q^2 \equiv |q|^2$ and $x^2 \equiv |x|^2$. The harmonics in (18) should now be chosen according to the general strategy outlined between equations (4) and (8). Thus in order to render convergent the sum (18), whose integral counterpart contains terms behaving like $\int_{|q|_{\min}}^{\infty} \mathrm{d}|q|\,|q|^{n-1}/|q|^{n-2}$, one again needs three subtraction terms. These are in fact nothing but the first three terms in the MacLaurin expansion of (21) for $|x| \ll |q|_{\min}$. This immediately leads to

$$\rho_n(x) = \frac{1}{(x^2)^{\frac{n-2}{2}}} + \sum \cdots \sum_{q\neq 0} \left\{ \frac{1}{|x-q|^{n-2}} \right.$$
$$\left. -\frac{1}{|q|^{n-2}} \left[1 + \frac{(n-2)}{q^2} \left(qx + \frac{1}{2q^2} \left(n(qx)^2 - q^2 x^2 \right) \right) \right] \right\} . \tag{22}$$

Again, absolute convergence is attained only by considering each term defined by the outermost parentheses as an indivisible unit. Hence the terms proportional to qx and to $1/2q^2$ cannot be summed separately (in which case they would appear to give zero by symmetry!) anymore than the ω^{-1} term in (2) can. Note that the subtraction terms are indeed harmonics as anticipated in (18).

This is perhaps an appropriate point to compare our results with Fueter's. Fueter introduces the unit quaternions $\boldsymbol{e}_\mu$ and $\bar{\boldsymbol{e}}_\mu$ corresponding to $(I, \boldsymbol{e}_i)$ and $(I, -\boldsymbol{e}_i)$, respectively. We have, as usual, $\boldsymbol{e}_i\boldsymbol{e}_j = -\delta_{ij} + \epsilon_{ijk}\boldsymbol{e}_k$, the indices i, j, k running from 1 to 3. Using these, one defines $\boldsymbol{x} = x_\mu \boldsymbol{e}_\mu$, $\bar{\boldsymbol{x}} = x_\mu \bar{\boldsymbol{e}}_\mu$, $D = \boldsymbol{e}_\mu \partial_\mu$ and $\bar{D} = \bar{\boldsymbol{e}}_\mu \partial_\mu$. One then has $x^2 = \bar{\boldsymbol{x}}\boldsymbol{x}$ and $\bar{D}D = \partial_\mu\partial_\mu$. Then, starting with the function

$$\boldsymbol{Z}(\boldsymbol{x}) = \frac{1}{\boldsymbol{x}} + \sum_{q\neq 0} \left\{ \frac{1}{\boldsymbol{x}-\boldsymbol{q}} + \frac{1}{\boldsymbol{q}} + \frac{1}{\boldsymbol{q}}\boldsymbol{x}\frac{1}{\boldsymbol{q}} + \frac{1}{\boldsymbol{q}}\boldsymbol{x}\frac{1}{\boldsymbol{q}}\boldsymbol{x}\frac{1}{\boldsymbol{q}} + \frac{1}{\boldsymbol{q}}\boldsymbol{x}\frac{1}{\boldsymbol{q}}\boldsymbol{x}\frac{1}{\boldsymbol{q}}\boldsymbol{x}\frac{1}{\boldsymbol{q}} \right\} \tag{23}$$

where $\boldsymbol{q} = n_a \boldsymbol{q}^{(a)} = n_a q^{(a)}_\mu \boldsymbol{e}_\mu$, the $q^{(a)}$ being lattice vectors, one arrives at the quaternionic version of Weierstrass's $\zeta(z)$ via

$$\zeta^{\boldsymbol{F}}(\boldsymbol{x}) = \partial_\mu\partial_\mu \boldsymbol{Z}(\boldsymbol{x})\,. \tag{24}$$

We will see later that (24) indeed transforms the same way as $\zeta(z)$ under lattice shifts. The function corresponding to $\ln\sigma(z)$ is

$$\rho(\boldsymbol{x}) = D\boldsymbol{Z}(\boldsymbol{x}) \tag{25}$$

and, remarkably, it turns out to have the quaternion-free form

$$\rho(x) = \frac{1}{x^2} + \sum_{q\neq 0} \left\{ \frac{1}{(x-q)^2} - \frac{1}{q^2} - \frac{2xq}{q^4} - \frac{1}{q^6}(4(qx)^2 - q^2x^2) \right\} \tag{26}$$

coinciding with the $n = 4$ case of (22).

Returning next to (12), we see that the analogue of $\zeta(z)$ is the n-gradient

$$\zeta^{(n)}_\mu(x) = \partial_\mu \rho_n(x)\,. \tag{27}$$

Taking $n = 4$ and contracting both sides with the quaternion units $\bar{\boldsymbol{e}}_\mu$, we recover the $\zeta^{\boldsymbol{F}}$ function of Fueter [2]

$$\zeta^{\boldsymbol{F}} = \bar{\boldsymbol{e}}_\mu\, \partial_\mu\, \rho_4 = \bar{D}\rho_4\,. \tag{28}$$

Leaving the quaternionic special case aside, let us now find the higher dimensional analogues of (8) and (9). Since

$$\partial_\mu\partial_\mu\, \rho_n = \partial_\mu\, \zeta^{(n)}_\mu = -\Omega_n \sum \cdots \sum \delta(x-q) \tag{29}$$

is a perfectly n-tuply periodic distribution, a shift by a lattice basis vector can only change $\zeta^{(n)}_\mu$ by a constant vector. Thus

$$\zeta^{(n)}_\mu(x + q^{(a)}) = \zeta^{(n)}_\mu(x) + \eta^{(n)(a)}_\mu\,. \tag{30}$$

This is indeed the transformation law for $\zeta^{\boldsymbol{F}}$ when $n = 4$. Putting $x = -q^{(a)}/2$ and noting that $\zeta^{(n)}_\mu(-x) = -\zeta^{(n)}_\mu(x)$, we find

$$2\zeta^{(n)}_\mu(q^{(a)}/2) = \eta^{(n)(a)}_\mu\,. \tag{31}$$

Integrating (30) and using $\rho_n(-x) = \rho_n(x)$, we obtain

$$\rho_n(x + q^{(a)}) = \rho_n(x) + \eta^{(n)(a)}(x + q^{(a)}/2) \tag{32}$$

where there is no sum on the index (a) on the right-hand side. Using equation (30), we can also obtain the higher dimensional form of Legendre's second relation (11). Let us evaluate $\oint_{\partial\mathcal{C}} \mathrm{d}\sigma_\mu\, \zeta^{(n)}_\mu(x)$, where $\partial\mathcal{C}$ is the surface of the fundamental hyperparallelepiped. Noting that the normal to the hyperplane defined by $\{q^{(a_2)}, \ldots, q^{(a_n)}\}$ is along the reciprocal vector $r^{(a_1)}$ given in (15), we find

$$V_n \sum_{a=1}^{n} \eta^{(n)(a)} r^{(a)} = \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{33}$$

as the n-dimensional generalization of Legendre's second relation.

There is an interesting relationship between ρ_n and ρ_{n-1} which can be iterated until one finally arrives at the Weierstrassian elliptic functions. It is simplest to describe this relationship in the case of a rectangular lattice although it is generally valid. Starting with ρ_n, one first chooses one lattice direction to coincide with, say, the x_n axis and then lets the lattice spacing (in the same direction only) become infinitesimal. Then, integrating over q_n, one obtains ρ_{n-1} up to a multiplicative constant. Repeating this procedure until ρ_3 is found, one finally has

$$-\int_{-\infty}^{\infty} \mathrm{d}q_3\, \rho_3 = \rho(z) + \rho(\bar{z}) = \int \mathrm{d}z\, \zeta(z) + \int \mathrm{d}\bar{z}\, \zeta(\bar{z}) \tag{34}$$

where the remaining lattice basis vectors $(\vec{q}^{\,(1)}, \vec{q}^{\,(2)})$ provide the periods (ω_1, ω_2). Thus the ρ_n presented here are indeed intimately connected with Weierstrassian functions. We also observe that the split into an analytic and anti-analytic part is a feature of two dimensions, not shared, for example, by Fueter's quaternionic functions.

Finally, the reader may wonder how the $\mathbf{R^n}$ counterpart of $\wp(z)$ is to be defined. It is clear that two derivatives of $\rho_n(x)$ will be involved. It is natural to classify the resulting functions according to their $SO(n)$ transformation properties. One has the symmetric traceless second-rank $SO(n)$ tensor

$$\wp^{(n)}_{\mu\nu} \equiv \left(\partial_\mu \partial_\nu - \frac{\delta_{\mu\nu}}{n} \partial_\lambda \partial_\lambda\right) \rho_n(x) \tag{35}$$

and the $SO(n)$ scalar

$$\pi^{(n)} \equiv \partial_\lambda \partial_\lambda\, \rho_n = -\Omega_n \sum \cdots \sum \delta(x-q) \tag{36}$$

already encountered in (29). Although both (35) and (36) are fully n-tuply periodic, the former involves true functions while the latter is an n-fold sum over distributions. $\wp^{(n)}_{\mu\nu}$ is also similar to $\wp(z)$ in that

$$\oint_{\partial\mathcal{C}} \mathrm{d}\sigma_\mu\, \wp^{(n)}_{\mu\nu} = \int_{\mathcal{C}} \mathrm{d}V_n\, \partial_\mu \wp^{(n)}_{\mu\nu} = 0 \tag{37}$$

just like $\wp(z)$ which obeys

$$\oint_{\partial\mathcal{C}} \wp(z)\, \mathrm{d}z = 0\,. \tag{38}$$

To summarize, we have constructed functions which: (i) reproduce in $\mathbf{R^n}$ the transformation properties under lattice shifts of the Weierstrassian functions, (ii) obey natural $\mathbf{R^n}$ generalizations of the two Legendre relations, (iii) reduce to the real parts of Weierstrassian functions upon an $(n-2)$-fold integration over lattice points, (iv) yield Fueter's earlier results (obtained by using quaternions) in $n = 4$. The generalization is based upon recognizing that $\ln \sigma(z)$ consists of a sum of Green's functions of the two-dimensional Laplacian over the lattice points, minus three harmonic subtraction terms per lattice point which serve to render the lattice sum convergent. As mentioned in the second paragraph, the $n = 2, 3, 4$ cases have already been used to model the non-perturbative ground states of Yang–Mills and Einstein theories as a 'foam' of topological solitons characteristic of those particular dimensions. The cases $6 \leqslant n \leqslant 22$ may prove relevant in superstring, Kaluza–Klein, supergravity, p-brane or bosonic string theories in which dimensions beyond 4 are compactified, effectively leading to the periodicity of functions in the corresponding coordinates.

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